ORTHOGONAL BASES OF SYMMETRIZED TENSOR SPACES

Randall R. Holmes

Abstract. It is shown that a symmetrized tensor space does not have an orthogonal basis consisting of standard symmetrized tensors if the associated permutation group is 2-transitive. In particular, no such basis exists if the group is the symmetric group or the alternating group as conjectured by T.-Y. Tam and the author.

Let V be a finite-dimensional complex inner product space and assume $m := \dim V \geq 2$ (to avoid trivialities). Let G be a subgroup of the symmetric group S_n. The n-fold tensor product $V^n = V \otimes \cdots \otimes V$ is a left $\mathbb{C}G$-module with action given by $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$ ($v_i \in V$, $\sigma \in G$).

Fix an orthonormal basis $\{e_1, \ldots, e_m\}$ of V. Let $\Gamma = \{\gamma \in \mathbb{Z}^n \mid 1 \leq \gamma_i \leq m\}$ and let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G. Given $\gamma \in \Gamma$ and $\chi \in \mathrm{Irr}(G)$, set $e^\chi_\gamma = s^\chi(e_{\gamma_1} \otimes \cdots \otimes e_{\gamma_n}) \in V^n$, where $s^\chi = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma \in \mathbb{C}G$. The vectors e^χ_γ are called *standard symmetrized tensors*. The inner product on V induces an inner product on V^n. If $W \leq V^n$ has a basis consisting of mutually orthogonal standard symmetrized tensors, we will say that W has an *o-basis*.

It follows easily from the standard results quoted below that if G is abelian, then V^n has an o-basis. In [2], it was shown that if G is the dihedral group $D_n \leq S_n$, then V^n has an o-basis if and only if n is a power of 2. In that paper, it was also shown that if $G = S_4$ or A_4 (alternating group), then V^n does not have an o-basis, and it was conjectured that in general this is the case whenever $G = S_n$ or A_n with $n \geq 4$. Here, we prove this conjecture by establishing the more general result that if G is 2-transitive and $n \geq 3$, then V^n does not have an o-basis.

We recall a few standard results. Choose a set Δ of representatives of the orbits of Γ under the right action of G given by $\gamma\tau = (\gamma_{\tau(1)}, \ldots, \gamma_{\tau(n)})$ ($\gamma \in \Gamma$, $\tau \in G$). Then $V^n = \bigoplus V^\chi_\gamma$ (orthogonal direct sum), where $V^\chi_\gamma := \langle e^\chi_{\gamma\tau} \mid \tau \in G \rangle$ and the sum is over all $\chi \in \mathrm{Irr}(G)$, $\gamma \in \Delta$.

Given $\gamma \in \Gamma$, set $G_\gamma := \{\sigma \in G \mid \gamma\sigma = \gamma\} \leq G$. We have

$$(e^\chi_{\gamma\mu}, e^\chi_{\gamma\tau}) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G_{\gamma\tau}} \chi(\sigma\tau^{-1}\mu) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\gamma} \chi(\sigma\mu\tau^{-1}),$$

the first equality from [1, p. 339] and the second from the observations that $\tau G_{\gamma\tau}\tau^{-1} = G_\gamma$ and $\chi(\sigma\tau^{-1}\mu) = \chi(\tau\sigma\tau^{-1}\mu\tau^{-1})$.

For any $H \leq G$, let $(\cdot,\cdot)_H$ denote the usual inner product on the space of complex-valued class functions on H. By [1, p. 339], $\dim V^\chi_\gamma = \chi(1)(\chi, 1)_{G_\gamma}$.

Set $I_n = \{1, \ldots, n\}$. Recall that G is 2-*transitive* if, with respect to the componentwise action, it is transitive on the set of pairs (i, j), with $i, j \in I_n$, $i \neq j$. Note that if G is 2-transitive, then for any $i \in I_n$, the subgroup $\{\sigma \in G \mid \sigma(i) = i\}$ of G is transitive on the set $I_n \backslash \{i\}$.

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Theorem. *Assume $n \geq 3$. If G is 2-transitive, then V^n does not have an o-basis.*

Proof. By the remarks above, it is enough to show that V_γ^χ does not have an o-basis for some $\chi \in \mathrm{Irr}(G)$, $\gamma \in \Gamma$.

Let $H = \{\sigma \in G \,|\, \sigma(n) = n\} < G$ and denote by ψ the induced character $(1_H)^G$, so that $\psi(\sigma) = |\{i \,|\, \sigma(i) = i\}|$ for $\sigma \in G$ (see [4, p. 68]).

Let $\rho \in G - H$ and for $i \in I_n$, set $R_i = \{\sigma \in H \,|\, \sigma\rho(i) = i\}$. Clearly, $R_i = \emptyset$ if $i \in \{n, \rho^{-1}(n)\}$. Assume $i \notin \{n, \rho^{-1}(n)\}$. Since H acts transitively on I_{n-1}, there exists some $\tau \in R_i$. Then $R_i = H_i\tau$, where $H_i := \{\sigma \in H \,|\, \sigma(i) = i\}$. Now $[H : H_i]$ equals the number of elements in the orbit of i under the action of H, so $[H : H_i] = n - 1$. Therefore, $|R_i| = |H_i\tau| = |H_i| = |H|/[H : H_i] = |H|/(n-1)$. We obtain the formula

$$\sum_{\sigma \in H} \psi(\sigma\rho) = \sum_{i=1}^{n} |R_i| = \sum_{i \neq n, \rho^{-1}(n)} |R_i| = \frac{n-2}{n-1}|H|.$$

Since $(\psi, 1)_G = (1, 1)_H = 1$ by Frobenius reciprocity, 1 is a constituent of ψ, whence $\chi := \psi - 1$ is a character of G. Moreover, the 2-transitivity of G implies that $(\psi, \psi)_G = 2$ (see [4, p. 68]). Hence, $(\chi, \chi)_G = 1$, so that χ is irreducible.

Let $\gamma = (1, \ldots, 1, 2) \in \Gamma$ and note that $G_\gamma = H$. Let μ and τ be representatives of distinct right cosets of H in G. Then $\rho := \mu\tau^{-1} \in G - H$, so the discussion above shows that

$$(e_{\gamma\mu}^\chi, e_{\gamma\tau}^\chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in H} \chi(\sigma\mu\tau^{-1}) = \frac{\chi(1)}{|G|} \left[\frac{n-2}{n-1}|H| - |H| \right] < 0.$$

It follows that distinct standard symmetrized tensors in V_γ^χ are not orthogonal.

On the other hand,

$$\dim V_\gamma^\chi = \chi(1)(\chi, 1)_H = (n-1)[(\psi, 1)_H - 1],$$

and since $(\psi, 1)_H = (\psi, \psi)_G = 2$ by Frobenius reciprocity, $\dim V_\gamma^\chi = n - 1 > 1$.

Therefore, V_γ^χ does not have an o-basis. This completes the proof. □

Corollary. *If $G = S_n$ $(n \geq 3)$ or $G = A_n$ $(n \geq 4)$, then V^n does not have an o-basis.*

Proof. Clearly each S_n is 2-transitive, and it is an easy exercise to show that A_n is 2-transitive if $n \geq 4$. □

2-transitive groups have been studied extensively (see [3, Chapter XII], for example).

References

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Department of Mathematics, Auburn University, AL 36849-5310